5.2 Integer Exponents and the Quotient Rule

OBJECTIVES

1. Use 0 as an exponent.
2. Use negative numbers as exponents.
3. Use the quotient rule for exponents.
4. Use combinations of rules.

In Section 5.1 we studied the product rule for exponents. In all our earlier work, exponents were positive integers. Now we want to develop meaning for exponents that are not positive integers.

Consider the following list of exponential expressions.

$$2^4 = 16$$
$$2^3 = 8$$
$$2^2 = 4$$

Do you see the pattern in the values? Each time we reduce the exponent by 1, the value is divided by 2 (the base). Using this pattern, we can continue the list to smaller and smaller integer exponents.

$$2^1 = 2$$
$$2^0 = 1$$
$$2^{-1} = \frac{1}{2}$$
$$2^{-2} = \frac{1}{4}$$
$$2^{-3} = \frac{1}{8}$$

From the preceding list, it appears that we should define 2^0 as 1 and negative exponents as reciprocals.

OBJECTIVE 1 Use 0 as an exponent. We want the definitions of 0 and negative exponents to satisfy the rules for exponents from Section 5.1. For example, if $6^0 = 1$,

$$6^0 \cdot 6^2 = 1 \cdot 6^2 = 6^2 \quad \text{and} \quad 6^0 \cdot 6^2 = 6^{0+2} = 6^2,$$

so the product rule is satisfied. Check that the power rules are also valid for a 0 exponent. Thus, we define a 0 exponent as follows.

Zero Exponent

For any nonzero real number a, $\quad a^0 = 1.$

Example: $17^0 = 1$

CAUTION Notice the difference between parts (b) and (c) of Example 1. In $(-60)^0$, the base is -60 and the exponent is 0. Any nonzero base raised to the 0 exponent is 1. But in -60^0, the base is 60. Then $60^0 = 1$, and $-60^0 = -1$.

OBJECTIVE 2 Use negative numbers as exponents. From the lists at the beginning of this section, since $2^{-2} = \frac{1}{4}$ and $2^{-3} = \frac{1}{8}$, we can deduce that 2^{-n} should equal $\frac{1}{2^n}$. Is the product rule valid in such cases? For example, if we multiply 6^{-2} by 6^2, we get

$$6^{-2} \cdot 6^2 = 6^{-2+2} = 6^0 = 1.$$

The expression 6^{-2} behaves as if it were the reciprocal of 6^2 because their product is 1. The reciprocal of 6^2 is also $\frac{1}{6^2}$, leading us to define 6^{-2} as $\frac{1}{6^2}$. This is a particular case of the definition of negative exponents.

Negative Exponents

For any nonzero real number a and any integer n, $a^{-n} = \frac{1}{a^n}$.

Example: $3^{-2} = \frac{1}{3^2}$

By definition, a^{-n} and a^n are reciprocals, since

$$a^n \cdot a^{-n} = a^n \cdot \frac{1}{a^n} = 1.$$

Since $1^n = 1$, the definition of a^{-n} can also be written

$$a^{-n} = \frac{1}{a^n} = \frac{1^n}{a^n} = \left(\frac{1}{a}\right)^n.$$

For example,

$$6^{-3} = \left(\frac{1}{6}\right)^3 \quad \text{and} \quad \left(\frac{1}{3}\right)^{-2} = 3^2.$$

EXAMPLE 2 Using Negative Exponents

Simplify by writing each expression with positive exponents.

(a) $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

(b) $5^{-3} = \frac{1}{5^3} = \frac{1}{125}$

(c) $\left(\frac{1}{2}\right)^{-3} = 2^3 = 8$ — $\frac{1}{2}$ and 2 are reciprocals.

Notice that we can change the base to its reciprocal if we also change the sign of the exponent.

(d) $\left(\frac{2}{5}\right)^{-4} = \left(\frac{5}{2}\right)^4$ — $\frac{2}{5}$ and $\frac{5}{2}$ are reciprocals.

(e) $\left(\frac{4}{3}\right)^{-5} = \left(\frac{3}{4}\right)^5$

(f) $4^{-1} - 2^{-1} = \frac{1}{4} - \frac{1}{2} = \frac{1}{4} - \frac{2}{4} = -\frac{1}{4}$

We apply the exponents first, then subtract.

(g) $p^{-2} = \frac{1}{p^2} \quad (p \neq 0)$

(h) $\frac{1}{x^{-4}} \quad (x \neq 0)$

$$\frac{1}{x^{-4}} = \frac{1^{-4}}{x^{-4}} \qquad 1^{-4} = 1$$

$$= \left(\frac{1}{x}\right)^{-4} \qquad \text{Power rule (c)}$$

$$= x^4 \qquad \tfrac{1}{x} \text{ and } x \text{ are reciprocals.}$$

Now Try Exercises 19, 21, 23, and 27.

CAUTION A negative exponent does not indicate a negative number; negative exponents lead to reciprocals.

Expression	Example	
a^{-n}	$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$	Not negative
$-a^{-n}$	$-3^{-2} = -\frac{1}{3^2} = -\frac{1}{9}$	Negative

The definition of negative exponents allows us to move factors across a fraction bar if we also change the signs of the exponents. For example,

$$\frac{2^{-3}}{3^{-4}} = \frac{\frac{1}{2^3}}{\frac{1}{3^4}} = \frac{1}{2^3} \cdot \frac{3^4}{1} = \frac{3^4}{2^3},$$

so

$$\frac{2^{-3}}{3^{-4}} = \frac{3^4}{2^3}.$$

Changing from Negative to Positive Exponents

For any nonzero numbers a and b, and any integers m and n,

$$\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m} \quad \text{and} \quad \left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m.$$

Examples: $\frac{3^{-5}}{2^{-4}} = \frac{2^4}{3^5}$ and $\left(\frac{4}{5}\right)^{-3} = \left(\frac{5}{4}\right)^3$

EXAMPLE 3 Changing from Negative to Positive Exponents

Write with only positive exponents. Assume all variables represent nonzero real numbers.

(a) $\frac{4^{-2}}{5^{-3}} = \frac{5^3}{4^2}$ or $\frac{125}{16}$ (b) $\frac{m^{-5}}{p^{-1}} = \frac{p^1}{m^5} = \frac{p}{m^5}$

(c) $\frac{a^{-2}b}{3d^{-3}} = \frac{bd^3}{3a^2}$

Notice that b in the numerator and 3 in the denominator were not affected.

(d) $x^3y^{-4} = \frac{x^3y^{-4}}{1} = \frac{x^3}{y^4}$ (e) $\left(\frac{x}{2y}\right)^{-4} = \left(\frac{2y}{x}\right)^4 = \frac{2^4y^4}{x^4}$

Now Try Exercises 31, 35, and 47.

CAUTION Be careful. We cannot change negative exponents to positive exponents using this rule if the exponents occur in a sum of terms. For example,

$$\frac{5^{-2} + 3^{-1}}{7 - 2^{-3}}$$

cannot be written with positive exponents using the rule given here. We would have to use the definition of a negative exponent to rewrite this expression with positive exponents, as

$$\frac{\frac{1}{5^2} + \frac{1}{3}}{7 - \frac{1}{2^3}}.$$

OBJECTIVE 3 Use the quotient rule for exponents. How should we handle the quotient of two exponential expressions with the same base? We know that

$$\frac{6^5}{6^3} = \frac{6 \cdot 6 \cdot 6 \cdot 6 \cdot 6}{6 \cdot 6 \cdot 6} = 6^2.$$

Notice that the difference between the exponents, $5 - 3 = 2$, is the exponent in the quotient. Also,

$$\frac{6^2}{6^4} = \frac{6 \cdot 6}{6 \cdot 6 \cdot 6 \cdot 6} = \frac{1}{6^2} = 6^{-2}.$$

Here, $2 - 4 = -2$. These examples suggest the **quotient rule for exponents.**

Quotient Rule for Exponents

For any nonzero real number a and any integers m and n,

$$\frac{a^m}{a^n} = a^{m-n}.$$

(Keep the base and subtract the exponents.)

Example: $\frac{5^8}{5^4} = 5^{8-4} = 5^4$

CAUTION A common **error** is to write $\frac{5^8}{5^4} = 1^{8-4} = 1^4$. Notice that by the quotient rule, the quotient should have the *same base,* 5. That is,

$$\frac{5^8}{5^4} = 5^{8-4} = 5^4.$$

If you are not sure, use the definition of an exponent to write out the factors:

$$5^8 = 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \quad \text{and} \quad 5^4 = 5 \cdot 5 \cdot 5 \cdot 5.$$

Then it is clear that the quotient is 5^4.

EXAMPLE 4 Using the Quotient Rule

Simplify, using the quotient rule for exponents. Write answers with positive exponents.

(a) $\frac{5^8}{5^6} = 5^{8-6} = 5^2$ or 25 (b) $\frac{4^2}{4^9} = 4^{2-9} = 4^{-7} = \frac{1}{4^7}$

(c) $\frac{5^{-3}}{5^{-7}} = 5^{-3-(-7)} = 5^4$ or 625 (d) $\frac{q^5}{q^{-3}} = q^{5-(-3)} = q^8 \quad (q \neq 0)$

(e) $\frac{3^2x^5}{3^4x^3} = \frac{3^2}{3^4} \cdot \frac{x^5}{x^3} = 3^{2-4} \cdot x^{5-3} = 3^{-2}x^2 = \frac{x^2}{3^2} \quad (x \neq 0)$

(f) $\frac{(m+n)^{-2}}{(m+n)^{-4}} = (m+n)^{-2-(-4)} = (m+n)^{-2+4} = (m+n)^2 \quad (m \neq -n)$

(g) $\frac{7x^{-3}y^2}{2^{-1}x^2y^{-5}} = \frac{7 \cdot 2^1y^2y^5}{x^2x^3} = \frac{14y^7}{x^5} \quad (x, y \neq 0)$

Now Try Exercises 29, 41, 45, and 49.

The definitions and rules for exponents given in this section and Section 5.1 are summarized here.

Definitions and Rules for Exponents

For any integers m and n:

		Examples
Product rule	$a^m \cdot a^n = a^{m+n}$	$7^4 \cdot 7^5 = 7^9$
Zero exponent	$a^0 = 1 \quad (a \neq 0)$	$(-3)^0 = 1$
Negative exponent	$a^{-n} = \frac{1}{a^n} \quad (a \neq 0)$	$5^{-3} = \frac{1}{5^3}$
Quotient rule	$\frac{a^m}{a^n} = a^{m-n} \quad (a \neq 0)$	$\frac{2^2}{2^5} = 2^{2-5} = 2^{-3} = \frac{1}{2^3}$
Power rules (a)	$(a^m)^n = a^{mn}$	$(4^2)^3 = 4^6$
(b)	$(ab)^m = a^mb^m$	$(3k)^4 = 3^4k^4$
(c)	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad (b \neq 0)$	$\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2}$

(continued)

Negative to positive rules $\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m}$ $(a \neq 0, b \neq 0)$ $\frac{2^{-4}}{5^{-3}} = \frac{5^3}{2^4}$

$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$ $\left(\frac{4}{7}\right)^{-2} = \left(\frac{7}{4}\right)^2$

OBJECTIVE 4 Use combinations of rules. We may sometimes need to use more than one rule to simplify an expression.

EXAMPLE 5 Using Combinations of Rules

Use the rules for exponents to simplify each expression. Assume all variables represent nonzero real numbers.

(a) $\frac{(4^2)^3}{4^5} = \frac{4^6}{4^5}$ Power rule (a)

$= 4^{6-5}$ Quotient rule

$= 4^1 = 4$

(b) $(2x)^3(2x)^2 = (2x)^5$ Product rule

$= 2^5x^5$ or $32x^5$ Power rule (b)

(c) $\left(\frac{2x^3}{5}\right)^{-4} = \left(\frac{5}{2x^3}\right)^4$ Negative to positive rule

$= \frac{5^4}{2^4x^{12}}$ or $\frac{625}{16x^{12}}$ Power rules (a)–(c)

(d) $\left(\frac{3x^{-2}}{4^{-1}y^3}\right)^{-3} = \frac{3^{-3}x^6}{4^3y^{-9}}$ Power rules (a)–(c)

$= \frac{x^6y^9}{4^3 \cdot 3^3}$ or $\frac{x^6y^9}{1728}$ Negative to positive rule

(e) $\frac{(4m)^{-3}}{(3m)^{-4}} = \frac{4^{-3}m^{-3}}{3^{-4}m^{-4}}$ Power rule (b)

$= \frac{3^4m^4}{4^3m^3}$ Negative to positive rule

$= \frac{3^4m^{4-3}}{4^3}$ Quotient rule

$= \frac{3^4m}{4^3}$ or $\frac{81m}{64}$

Now Try Exercises 57, 63, 65, and 67.

NOTE Since the steps can be done in several different orders, there are many equally correct ways to simplify expressions like those in Examples 5(d) and 5(e).

5.2 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 9/Videotape 8

Each expression is either equal to 0, 1, or −1. Decide which is correct. See Example 1.

1. 9^0 **2.** 5^0 **3.** $(-4)^0$ **4.** $(-10)^0$

5. -9^0 **6.** -5^0 **7.** $(-2)^0 - 2^0$ **8.** $(-8)^0 - 8^0$

9. $\frac{0^{10}}{10^0}$ **10.** $\frac{0^5}{5^0}$

Match each expression in Column I with the equivalent expression in Column II. Choices in Column II may be used once, more than once, or not at all.

I	II
11. -2^{-4}	**A.** 8
12. $(-2)^{-4}$	**B.** 16
13. 2^{-4}	**C.** $-\frac{1}{16}$
14. $\frac{1}{2^{-4}}$	**D.** −8
15. $\frac{1}{-2^{-4}}$	**E.** −16
16. $\frac{1}{(-2)^{-4}}$	**F.** $\frac{1}{16}$

Evaluate each expression. See Examples 1 and 2.

17. $7^0 + 9^0$ **18.** $8^0 + 6^0$ **19.** 4^{-3} **20.** 5^{-4}

21. $\left(\frac{1}{2}\right)^{-4}$ **22.** $\left(\frac{1}{3}\right)^{-3}$ **23.** $\left(\frac{6}{7}\right)^{-2}$ **24.** $\left(\frac{2}{3}\right)^{-3}$

25. $(-3)^{-4}$ **26.** $(-4)^{-3}$ **27.** $5^{-1} + 3^{-1}$ **28.** $6^{-1} + 2^{-1}$

Use the quotient rule to simplify each expression. Write the expression with positive exponents. Assume that all variables represent nonzero real numbers. See Examples 2–4.

29. $\frac{5^8}{5^5}$ **30.** $\frac{11^6}{11^3}$ **31.** $\frac{3^{-2}}{5^{-3}}$ **32.** $\frac{4^{-3}}{3^{-2}}$ **33.** $\frac{5}{5^{-1}}$

34. $\frac{6}{6^{-2}}$ **35.** $\frac{x^{12}}{x^{-3}}$ **36.** $\frac{y^4}{y^{-6}}$ **37.** $\frac{1}{6^{-3}}$ **38.** $\frac{1}{5^{-2}}$

39. $\frac{2}{r^{-4}}$ **40.** $\frac{3}{s^{-8}}$ **41.** $\frac{4^{-3}}{5^{-2}}$ **42.** $\frac{6^{-2}}{5^{-4}}$ **43.** p^5q^{-8}

44. $x^{-8}y^4$ **45.** $\frac{r^5}{r^{-4}}$ **46.** $\frac{a^6}{a^{-4}}$ **47.** $\frac{x^{-3}y}{4z^{-2}}$ **48.** $\frac{p^{-5}q^{-4}}{9r^{-3}}$

49. $\frac{(a+b)^{-3}}{(a+b)^{-4}}$ **50.** $\frac{(x+y)^{-8}}{(x+y)^{-9}}$ **51.** $\frac{(x+2y)^{-3}}{(x+2y)^{-5}}$ **52.** $\frac{(p-3q)^{-2}}{(p-3q)^{-4}}$

RELATING CONCEPTS (EXERCISES 53–56)

For Individual or Group Work

In Objective 1, we showed how 6^0 acts as 1 when it is applied to the product rule, thus motivating the definition for 0 as an exponent. We can also use the quotient rule to motivate this definition. ***Work Exercises 53–56 in order.***

53. Consider the expression $\frac{25}{25}$. What is its simplest form?

54. Because $25 = 5^2$, the expression $\frac{25}{25}$ can be written as the quotient of powers of 5. Write the expression in this way.

55. Apply the quotient rule for exponents to the expression you wrote in Exercise 54. Give the answer as a power of 5.

56. Your answers in Exercises 53 and 55 must be equal because they both represent $\frac{25}{25}$. Write this equality. What definition does this result support?

Use a combination of the rules for exponents to simplify each expression. Write answers with only positive exponents. Assume that all variables represent nonzero real numbers. See Example 5.

57. $\frac{(7^4)^3}{7^9}$ **58.** $\frac{(5^3)^2}{5^2}$ **59.** $x^{-3} \cdot x^5 \cdot x^{-4}$ **60.** $y^{-8} \cdot y^5 \cdot y^{-2}$

61. $\frac{(3x)^{-2}}{(4x)^{-3}}$ **62.** $\frac{(2y)^{-3}}{(5y)^{-4}}$ **63.** $\left(\frac{x^{-1}y}{z^2}\right)^{-2}$ **64.** $\left(\frac{p^{-4}q}{r^{-3}}\right)^{-3}$

65. $(6x)^4(6x)^{-3}$ **66.** $(10y)^9(10y)^{-8}$ **67.** $\frac{(m^7n)^{-2}}{m^{-4}n^3}$ **68.** $\frac{(m^8n^{-4})^2}{m^{-2}n^5}$

69. $\frac{(x^{-1}y^2z)^{-2}}{(x^{-3}y^3z)^{-1}}$ **70.** $\frac{(a^{-2}b^{-3}c^{-4})^{-5}}{(a^2b^3c^4)^5}$ **71.** $\left(\frac{xy^{-2}}{x^2y}\right)^{-3}$ **72.** $\left(\frac{wz^{-5}}{w^{-3}z}\right)^{-2}$

73. $\frac{(4a^2b^3)^{-2}(2ab^{-1})^3}{(a^3b)^{-4}}$ **74.** $\frac{(m^6n)^{-2}(m^2n^{-2})^3}{m^{-1}n^{-2}}$

75. $\frac{(2y^{-1}z^2)^2(3y^{-2}z^{-3})^3}{(y^3z^2)^{-1}}$ **76.** $\frac{(3p^{-2}q^3)^2(5p^{-1}q^{-4})^{-1}}{(p^2q^{-2})^{-3}}$

77. $\frac{(9^{-1}z^{-2}x)^{-1}(4z^2x^4)^{-2}}{(5z^{-2}x^{-3})^2}$ **78.** $\frac{(4^{-1}a^{-1}b^{-2})^{-2}(5a^{-3}b^4)^{-2}}{(3a^{-3}b^{-5})^2}$

79. Consider the following typical student **error:**

$$\frac{16^3}{2^2} = \left(\frac{16}{2}\right)^{3-2} = 8^1 = 8.$$

Explain what the student did incorrectly, and then give the correct answer.

80. Consider the following typical student **error:**

$$-5^4 = (-5)^4 = 625.$$

Explain what the student did incorrectly, and then give the correct answer.

SUMMARY EXERCISES ON THE RULES FOR EXPONENTS

Use the rules for exponents to simplify each expression. Use only positive exponents in your answers. Assume all variable expressions represent positive numbers.

1. $\left(\frac{6x^2}{5}\right)^{12}$ **2.** $\left(\frac{rs^2t^3}{3t^4}\right)^6$ **3.** $(10x^2y^4)^2(10xy^2)^3$

4. $(-2ab^3c)^4(-2a^2b)^3$ **5.** $\left(\frac{9wx^3}{y^4}\right)^3$ **6.** $(4x^{-2}y^{-3})^{-2}$

7. $\frac{c^{11}(c^2)^4}{(c^3)^3(c^2)^{-6}}$ **8.** $\left(\frac{k^4t^2}{k^2t^{-4}}\right)^{-2}$ **9.** $5^{-1} + 6^{-1}$

10. $\frac{(3y^{-1}z^3)^{-1}(3y^2)}{(y^3z^2)^{-3}}$ **11.** $\frac{(2xy^{-1})^3}{2^3x^{-3}y^2}$ **12.** $-8^0 + (-8)^0$

13. $(z^4)^{-3}(z^{-2})^{-5}$ **14.** $\left(\frac{r^2st^5}{3r}\right)^{-2}$ **15.** $\frac{(3^{-1}x^{-3}y)^{-1}(2x^2y^{-3})^2}{(5x^{-2}y^2)^{-2}}$

16. $\left(\frac{5x^2}{3x^{-4}}\right)^{-1}$ **17.** $\left(\frac{-2x^{-2}}{2x^2}\right)^{-2}$ **18.** $\frac{(x^{-4}y^2)^3(x^2y)^{-1}}{(xy^2)^{-3}}$

19. $\frac{(a^{-2}b^3)^{-4}}{(a^{-3}b^2)^{-2}(ab)^{-4}}$ **20.** $(2a^{-30}b^{-29})(3a^{31}b^{30})$ **21.** $5^{-2} + 6^{-2}$

22. $\left[\frac{(x^{47}y^{23})^2}{x^{-26}y^{-42}}\right]^0$ **23.** $\left(\frac{7a^2b^3}{2}\right)^3$ **24.** $-(-12^0)$

25. $-(-12)^0$ **26.** $\frac{0^{12}}{12^0}$ **27.** $\frac{(2xy^{-3})^{-2}}{(3x^{-2}y^4)^{-3}}$

28. $\left(\frac{a^2b^3c^4}{a^{-2}b^{-3}c^{-4}}\right)^{-2}$ **29.** $(6x^{-5}z^3)^{-3}$ **30.** $(2p^{-2}qr^{-3})(2p)^{-4}$

31. $\frac{(xy)^{-3}(xy)^5}{(xy)^{-4}}$ **32.** $42^0 - (-12)^0$ **33.** $\frac{(7^{-1}x^{-3})^{-2}(x^4)^{-6}}{7^{-1}x^{-3}}$

34. $\left(\frac{3^{-4}x^{-3}}{3^{-3}x^{-6}}\right)^{-2}$ **35.** $(5p^{-2}q)^{-3}(5pq^3)^4$ **36.** $8^{-1} + 6^{-1}$

37. $\left[\frac{4r^{-6}s^{-2}t}{2r^8s^{-4}t^2}\right]^{-1}$ **38.** $(13x^{-6}y)(13x^{-6}y)^{-1}$ **39.** $\frac{(8pq^{-2})^4}{(8p^{-2}q^{-3})^3}$

40. $\left(\frac{mn^{-2}p}{m^2np^4}\right)^{-2}\left(\frac{mn^{-2}p}{m^2np^4}\right)^3$ **41.** $-(-3^0)^0$ **42.** $5^{-1} - 8^{-1}$