5.1 The Product Rule and Power Rules for Exponents

OBJECTIVES

1. Identify bases and exponents.
2. Use the product rule for exponents.
3. Use the rule $(a^m)^n = a^{mn}$.
4. Use the rule $(ab)^m = a^m b^m$.
5. Use the rule $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.
6. Use combinations of rules.

OBJECTIVE 1 Identify bases and exponents. Recall from Section 1.2 that in the expression 5^2, the number 5 is the *base* and 2 is the *exponent* or *power*. The expression 5^2 is called an *exponential expression*. Usually we do not write the exponent when it is 1; however, sometimes it is convenient to do so. In general, for any quantity a, $a^1 = a$.

EXAMPLE 1 Determining the Base and Exponent in Exponential Expressions

Evaluate each exponential expression. Name the base and the exponent.

	Base	*Exponent*
(a) $5^4 = 5 \cdot 5 \cdot 5 \cdot 5 = 625$	5	4
(b) $-5^4 = -1 \cdot 5^4 = -1 \cdot (5 \cdot 5 \cdot 5 \cdot 5) = -625$	5	4
(c) $(-5)^4 = (-5)(-5)(-5)(-5) = 625$	-5	4

Now Try Exercises 15 and 17.

CAUTION Note the differences between parts (b) and (c) of Example 1. In -5^4 the lack of parentheses shows that the exponent 4 refers only to the base 5, not -5; in $(-5)^4$ the parentheses show that the exponent 4 refers to the base -5. In summary, $-a^n$ and $(-a)^n$ are not necessarily the same.

Expression	*Base*	*Exponent*	*Example*
$-a^n$	a	n	$-3^2 = -(3 \cdot 3) = -9$
$(-a)^n$	$-a$	n	$(-3)^2 = (-3)(-3) = 9$

OBJECTIVE 2 Use the product rule for exponents. By the definition of exponents,

$$\begin{aligned} 2^4 \cdot 2^3 &= \underbrace{(2 \cdot 2 \cdot 2 \cdot 2)}_{\text{4 factors}}\underbrace{(2 \cdot 2 \cdot 2)}_{\text{3 factors}} \\ &= \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{4 + 3 = 7 \text{ factors}} \\ &= 2^7. \end{aligned}$$

Also,

$$\begin{aligned} 6^2 \cdot 6^3 &= (6 \cdot 6)(6 \cdot 6 \cdot 6) \\ &= 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \\ &= 6^5. \end{aligned}$$

Generalizing from these examples, $2^4 \cdot 2^3 = 2^{4+3} = 2^7$ and $6^2 \cdot 6^3 = 6^{2+3} = 6^5$, suggests the **product rule for exponents.**

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Recall from Section 1.2 that in the ex- ... the *exponent* or *power*. The expression ... ally we do not write the exponent when it ... to do so. In general, for any quantity a,

...nd Exponent in Exponential

... the base and the exponent.

	Base	Exponent
	5	4
625	5	4
	5	4

Now Try Exercises 15 and 17.

... (b) and (c) of Example 1. In ... nent 4 refers only to the base 5, ... onent 4 refers to the base -5. ... same.

... definition of exponents,

Product Rule for Exponents

For any positive integers m and n, $a^m \cdot a^n = a^{m+n}$.
(Keep the same base; add the exponents.)
Example: $6^2 \cdot 6^5 = 6^{2+5} = 6^7$

CAUTION Avoid the common error of multiplying the bases when using the product rule:

$$6^2 \cdot 6^5 \neq 36^7.$$

Keep the same base and add the exponents.

EXAMPLE 2 Using the Product Rule

Use the product rule for exponents to find each result when possible.

(a) $6^3 \cdot 6^5 = 6^{3+5} = 6^8$

(b) $(-4)^7(-4)^2 = (-4)^{7+2} = (-4)^9$

(c) $x^2 \cdot x = x^2 \cdot x^1 = x^{2+1} = x^3$

(d) $m^4m^3m^5 = m^{4+3+5} = m^{12}$

(e) $2^3 \cdot 3^2$

The product rule does not apply to the product $2^3 \cdot 3^2$, since the bases are different.

$$2^3 \cdot 3^2 = 8 \cdot 9 = 72$$

(f) $2^3 + 2^4$

The product rule does not apply to $2^3 + 2^4$, since this is a *sum*, not a *product*.

$$2^3 + 2^4 = 8 + 16 = 24$$

(g) $(2x^3)(3x^7)$

Since $2x^3$ means $2 \cdot x^3$ and $3x^7$ means $3 \cdot x^7$, we use the associative and commutative properties to get

$$(2x^3)(3x^7) = (2 \cdot 3) \cdot (x^3 \cdot x^7) = 6x^{10}.$$

Now Try Exercises 25, 29, 31, 35, and 39.

CAUTION Be sure you understand the difference between *adding* and *multiplying* exponential expressions. For example,

$$8x^3 + 5x^3 = (8 + 5)x^3 = 13x^3,$$

but $$(8x^3)(5x^3) = (8 \cdot 5)x^{3+3} = 40x^6.$$

OBJECTIVE 3 Use the rule $(a^m)^n = a^{mn}$. We simplify an expression such as $(8^3)^2$ with the product rule for exponents.

$$(8^3)^2 = (8^3)(8^3) = 8^{3+3} = 8^6$$

Power Rule (a) for Exponents

For any positive integers m and n, $(a^m)^n = a^{mn}$.
(Raise a power to a power by multiplying exponents.)
Example: $(3^2)^4 = 3^{2 \cdot 4} = 3^8$

EXAMPLE 3 Using Power Rule (a)

Use power rule (a) for exponents to simplify each expression.

(a) $(2^5)^3 = 2^{5 \cdot 3} = 2^{15}$ **(b)** $(5^7)^2 = 5^{7(2)} = 5^{14}$ **(c)** $(x^2)^5 = x^{2(5)} = x^{10}$

Now Try Exercises 43 and 45.

OBJECTIVE 4 Use the rule $(ab)^m = a^m b^m$. We can use the properties studied in Chapter 1 to develop two more rules for exponents. Using the definition of an exponential expression and the commutative and associative properties, we can rewrite the expression $(4x)^3$ as follows.

$$(4x)^3 = (4x)(4x)(4x) \quad \text{Definition of exponent}$$
$$= (4 \cdot 4 \cdot 4)(x \cdot x \cdot x) \quad \text{Commutative and associative properties}$$
$$= 4^3 \cdot x^3 \quad \text{Definition of exponent}$$

This example suggests **power rule (b) for exponents.**

Power Rule (b) for Exponents

For any positive integer m, $(ab)^m = a^m b^m$.
(Raise a product to a power by raising each factor to the power.)
Example: $(2p)^5 = 2^5 p^5$

EXAMPLE 4 Using Power Rule (b)

Use power rule (b) for exponents to simplify each expression.

(a) $(3xy)^2 = 3^2x^2y^2 = 9x^2y^2$ Power rule (a)

(b) $5(pq)^2 = 5(p^2q^2)$ Power rule (b)
$= 5p^2q^2$ Multiply.

(c) $3(2m^2p^3)^4 = 3[2^4(m^2)^4(p^3)^4]$ Power rule (b)
$= 3 \cdot 2^4 m^8 p^{12}$ Power rule (a)
$= 48m^8p^{12}$ Multiply.

(d) $(-5^6)^3 = (-1 \cdot 5^6)^3$
$= (-1)^3 \cdot (5^6)^3$
$= -1 \cdot 5^{18}$
$= -5^{18}$

Now Try Exercises 49 and 53.

CAUTION Power rule (b) *does not* apply to a *sum.*

$$(x + 4)^2 \neq x^2 + 4^2$$

OBJECTIVE 5 Use the rule $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$. Since the quotient $\frac{a}{b}$ can be written as $a\left(\frac{1}{b}\right)$, we use power rule (b), together with some of the properties of real numbers, to get **power rule (c) for exponents.**

Power Rule (c) for Exponents

For any positive integer m,

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad (b \neq 0).$$

(Raise a quotient to a power by raising both numerator and denominator to the power.)

Example: $\left(\frac{5}{3}\right)^2 = \frac{5^2}{3^2}$

EXAMPLE 5 Using Power Rule (c)

Use power rule (c) for exponents to simplify each expression.

(a) $\left(\frac{2}{3}\right)^5 = \frac{2^5}{3^5}$ **(b)** $\left(\frac{m}{n}\right)^3 = \frac{m^3}{n^3} \quad (n \neq 0)$

Now Try Exercises 59 and 61.

In the following box, we list the rules for exponents discussed in this section. These rules are basic to the study of algebra.

Rules for Exponents

For positive integers m and n:

		Examples
Product rule	$a^m \cdot a^n = a^{m+n}$	$6^2 \cdot 6^5 = 6^{2+5} = 6^7$
Power rules (a)	$(a^m)^n = a^{mn}$	$(3^2)^4 = 3^{2 \cdot 4} = 3^8$
(b)	$(ab)^m = a^m b^m$	$(2p)^5 = 2^5 p^5$
(c)	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad (b \neq 0)$	$\left(\frac{5}{3}\right)^2 = \frac{5^2}{3^2}$

OBJECTIVE 6 Use combinations of rules. As shown in the next example, more than one rule may be needed to simplify an expression with exponents.

EXAMPLE 6 Using Combinations of Rules

Use the rules for exponents to simplify each expression.

(a) $\left(\frac{2}{3}\right)^2 \cdot 2^3 = \frac{2^2}{3^2} \cdot \frac{2^3}{1}$ Power rule (c)

$= \frac{2^2 \cdot 2^3}{3^2 \cdot 1}$ Multiply fractions.

$= \frac{2^5}{3^2}$ Product rule

(b) $(5x)^3(5x)^4 = (5x)^7$ Product rule

$= 5^7x^7$ Power rule (b)

(c) $(2x^2y^3)^4(3xy^2)^3 = 2^4(x^2)^4(y^3)^4 \cdot 3^3x^3(y^2)^3$ Power rule (b)

$= 2^4x^8y^{12} \cdot 3^3x^3y^6$ Power rule (a)

$= 2^4 \cdot 3^3x^8x^3y^{12}y^6$ Commutative and associative properties

$= 16 \cdot 27x^{11}y^{18}$ Product rule

$= 432x^{11}y^{18}$

(d) $(-x^3y)^2(-x^5y^4)^3 = (-1 \cdot x^3y)^2(-1 \cdot x^5y^4)^3$

$= (-1)^2(x^3)^2y^2 \cdot (-1)^3(x^5)^3(y^4)^3$ Power rule (b)

$= (-1)^2(x^6)(y^2)(-1)^3(x^{15})(y^{12})$ Power rule (a)

$= (-1)^5(x^{21})(y^{14})$ Product rule

$= -x^{21}y^{14}$

Now Try Exercises 63, 67, 75, and 77.

CAUTION Refer to Example 6(c). Notice that

$$(2x^2y^3)^4 = 2^4x^{2\cdot4}y^{3\cdot4}, \quad \text{not} \quad (2 \cdot 4)x^{2\cdot4}y^{3\cdot4}.$$

Do not multiply the coefficient 2 and the exponent 4.

5.1 EXERCISES

For Extra Help
- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 9/Videotape 8

Decide whether each statement is true *or* false.

1. $3^3 = 9$ **2.** $(-2)^4 = 2^4$ **3.** $(a^2)^3 = a^5$ **4.** $\left(\frac{1}{4}\right)^2 = \frac{1}{4^2}$

Write each expression using exponents.

5. $w \cdot w \cdot w \cdot w \cdot w \cdot w$ **6.** $t \cdot t \cdot t \cdot t \cdot t \cdot t \cdot t$

7. $\frac{1}{4 \cdot 4 \cdot 4 \cdot 4}$ **8.** $\frac{1}{3 \cdot 3 \cdot 3}$

9. $(-7x)(-7x)(-7x)(-7x)$ **10.** $(-8p)(-8p)$

11. $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$ **12.** $\left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)$

13. Explain how the expressions $(-3)^4$ and -3^4 are different.

14. Explain how the expressions $(5x)^3$ and $5x^3$ are different.

Identify the base and the exponent for each exponential expression. In Exercises 15–18, also evaluate each expression. See Example 1.

15. 3^5
16. 2^7
17. $(-3)^5$
18. $(-2)^7$
19. $(-6x)^4$
20. $(-8x)^4$
21. $-6x^4$
22. $-8x^4$

23. Explain why the product rule does not apply to the expression $5^2 + 5^3$. Then evaluate the expression by finding the individual powers and adding the results.

24. Repeat Exercise 23 for the expression $(-4)^3 + (-4)^4$.

Use the product rule, if possible, to simplify each expression. Write each answer in exponential form. See Example 2.

25. $5^2 \cdot 5^6$
26. $3^6 \cdot 3^7$
27. $4^2 \cdot 4^7 \cdot 4^3$
28. $5^3 \cdot 5^8 \cdot 5^2$
29. $(-7)^3(-7)^6$
30. $(-9)^8(-9)^5$
31. $t^3 \cdot t^8 \cdot t^{13}$
32. $n^5 \cdot n^6 \cdot n^9$
33. $(-8r^4)(7r^3)$
34. $(10a^7)(-4a^3)$
35. $(-6p^5)(-7p^5)$
36. $(-5w^8)(-9w^8)$
37. $(5x^2)(-2x^3)(3x^4)$
38. $(12y^3)(4y)(-3y^5)$
39. $3^8 + 3^9$
40. $4^{12} + 4^5$
41. $5^8 \cdot 3^9$
42. $6^3 \cdot 8^9$

Use the power rules for exponents to simplify each expression. Write each answer in exponential form. See Examples 3–5.

43. $(4^3)^2$
44. $(8^3)^6$
45. $(t^4)^5$
46. $(y^6)^5$
47. $(7r)^3$
48. $(11x)^4$
49. $(5xy)^5$
50. $(9pq)^6$
51. $(-5^2)^6$
52. $(-9^4)^8$
53. $(-8^3)^5$
54. $(-7^5)^7$
55. $8(qr)^3$
56. $4(vw)^5$
57. $\left(\frac{1}{2}\right)^3$
58. $\left(\frac{1}{3}\right)^5$
59. $\left(\frac{a}{b}\right)^3 \quad (b \neq 0)$
60. $\left(\frac{r}{t}\right)^4 \quad (t \neq 0)$
61. $\left(\frac{9}{5}\right)^8$
62. $\left(\frac{12}{7}\right)^3$

Use a combination of the rules of exponents to simplify each expression. See Example 6.

63. $\left(\frac{5}{2}\right)^3 \cdot \left(\frac{5}{2}\right)^2$
64. $\left(\frac{3}{4}\right)^5 \cdot \left(\frac{3}{4}\right)^6$
65. $\left(\frac{9}{8}\right)^3 \cdot 9^2$
66. $\left(\frac{8}{5}\right)^4 \cdot 8^3$
67. $(2x)^9(2x)^3$
68. $(6y)^5(6y)^8$
69. $(-6p)^4(-6p)$
70. $(-13q)^3(-13q)$
71. $(6x^2y^3)^5$
72. $(5r^5t^6)^7$
73. $(x^2)^3(x^3)^5$
74. $(y^4)^5(y^3)^5$
75. $(2w^2x^3y)^2(x^4y)^5$
76. $(3x^4y^2z)^3(yz^4)^5$
77. $(-r^4s)^2(-r^2s^3)^5$

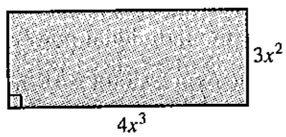
78. $(-ts^6)^4(-t^3s^5)^3$ **79.** $\left(\frac{5a^2b^5}{c^6}\right)^3$ $(c \neq 0)$ **80.** $\left(\frac{6x^3y^9}{z^5}\right)^4$ $(z \neq 0)$

81. A student tried to simplify $(10^2)^3$ as 1000^6. Is this correct? If not, how is it simplified using the product rule for exponents?

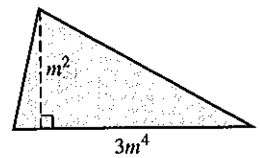
82. Explain why $(3x^2y^3)^4$ is *not* equivalent to $(3 \cdot 4)x^8y^{12}$.

Find the area of each figure. Use the formulas found on the inside covers. (The small squares in the figures indicate 90° *right angles.)*

83.



84.



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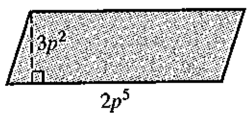

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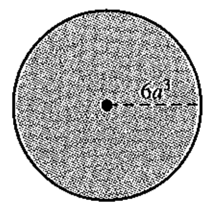
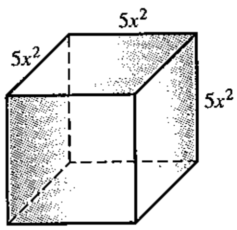


Find the volume of each figure. Use the formulas found on the inside covers.

87.



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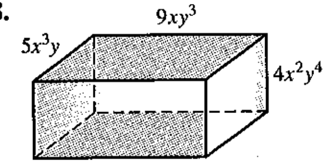


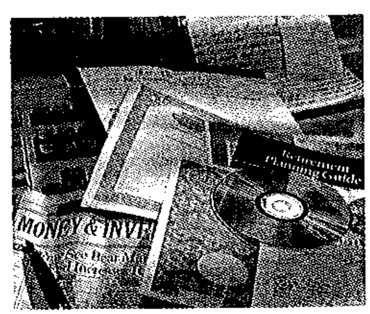
89. Assume a is a positive number greater than 1. Arrange the following terms in order from smallest to largest: $-(-a)^3$, $-a^3$, $(-a)^4$, $-a^4$. Explain how you decided on the order.

90. Devise a rule to tell whether an exponential expression with a negative base is positive or negative.

In Chapter 2 we used the formula for simple interest, $I = prt$, which deals with interest paid only on the principal. With **compound interest,** interest is paid on the principal and the interest earned earlier. The formula for compound interest, which involves an exponential expression, is

$$A = P(1 + r)^n.$$

Here A is the amount accumulated from a principal of P dollars left untouched for n years with an annual interest rate r (expressed as a decimal).



In Exercises 91–94, use this formula and a calculator to find A to the nearest cent.

91. $P = \$250$, $r = .04$, $n = 5$

92. $P = \$400$, $r = .04$, $n = 3$

93. $P = \$1500$, $r = .035$, $n = 6$

94. $P = \$2000$, $r = .025$, $n = 4$